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Algorithmic symmetry classification with invariance

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Abstract Symmetry classification for a system of differential equations can be achieved algorithmically by applying a differential reduction and completion algorithm to the infinitesimal determining equations of the system. The branches of the classification should be invariant under the action of the equivalence group. We show that such invariance can be tested algorithmically knowing only the determining equations of the equivalence group. The method relies on computing the prolongation of a group operator reduced modulo these determining equations. The method is implemented in Maple: a novel pivot selection strategy is able to guide the `rifsimp` command towards more favourable branchings.

Keywords Differential equations · Differential reduction · Equivalence transformation · Symmetry classification

1 Introduction

Symmetry methods for partial differential equations (PDEs) are one of the outstanding tools in the arsenal of modern applied mathematics. Analysing a PDE system E for its symmetries is by now a widely known process, complete with textbooks [1–3] and computer packages [4–6]. In this paper we shall be concerned with *symmetry classification* for PDEs, which is more ambitious. Rather than a specific PDE E, one attempts to deal with a whole family C of them at once. The family C is parametrised by certain constants or functions, which we collectively call *arbitrary elements*. The problem is to describe the symmetries of each member of the class. By doing so, one hopes to identify assignments of arbitrary elements that are physically realistic and have symmetries that permit solving useful boundary-value problems.

Some hundreds of pages of classifications for PDEs are collected in [7], and many more DE systems have subsequently been classified [8–11]. Such problems have been dealt with by a variety of methods, starting with Lie [12], and based on analysis of the infinitesimal determining equations for the symmetries. Ovsiannikov [3] further developed Lie's method, so that it is sometimes described as the 'Lie–Ovsiannikov' method [9]. Other more

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geometric methods exist [8, 13, 14], [15, Chap. 10], but there is no single dominant method: the candidates suffer variously from not being geometric, or not algorithmic, or being algorithmic but overwhelmed by expression swell on difficult problems, or of producing only partial results. In this paper we shall be concerned with an algorithmic enhancement of the Lie–Ovsiannikov infinitesimal method.

An approach that has met some success is to apply a differential reduction and completion (DRC) algorithm to the symmetry determining system $\mathcal{S}$. This was described by Reid [16] and applied by him using his *rif* algorithm ('reduced involutive form'). However, other methods such as the Ritt–Wu method or the Rosenfeld–Gröbner algorithm of Boulier, et al. [17] can equally well be used. A drawback of such methods is that expression swell can overwhelm the calculation. A second, related issue is that the method takes no account of the transformation properties of the PDEs.

Much of the algebraic complexity arising during DRC is due to degrees of freedom present due to the action of the 'equivalence group' Γ of the class C of PDE. This is a group Γ of point transformations that maps members of C to members of C [3, Sect. 6.4], [18–20]. The equivalence group can be used for parameter removal after classifying symmetries. Throughout the *CRC Handbook*, Ibragimov [7, Part B] uses the equivalence group in this way to clarify and simplify group analyses. Many other authors also use the equivalence group action to assist in symmetry classification [8,13,21].

The equivalence group action on arbitrary elements naturally partitions them into equivalence classes. Two equations $E_1, E_2 \in C$ connected by a transformation from Γ have symmetry groups that differ only by a change of coordinates. When classifying symmetries it is therefore natural to insist that equations that are Γ-equivalent should be in the same grouping in the symmetry classification.

Reid's method [16] when applied to symmetry classification can perform well on moderately complex classifications (perhaps two or three arbitrary functions of a single variable). However, because the method knows nothing of the equivalence group Γ, it can sometimes split into classes that are not Γ-invariant. The `rifsimp` implementation in Maple [22] allows different preferences for desirable case splits, such as smallest 'pivot' expression, smallest equation, and the like. However, in a symmetry classification, such considerations should be subordinate to the desired Γ-invariance.

One resolution of this is to perform the entire symmetry calculation in a Γ-invariant way. Lisle and Reid [10] describe one such method for achieving this: the symmetry operators are written with respect to a Γ-invariant basis of (non-commuting) differential operators, casting the determining equations into Γ-invariant form. All subsequent DRC steps are automatically invariant, and a good deal of the algebraic complexity evaporates. Following Fels and Olver [23,24], there are by now quite slick mechanisms for carrying out geometric calculations in an invariant way, and invariant forms of DRC algorithm have been developed [25,26].

The starting point for such invariant methods is an explicit parametrisation of the equivalence group, and this may not be available algorithmically. In contrast Reid's (non-invariant) method is completely algorithmic, uses computer algebra tools that are by now standard, and requires little guidance from the user. There is hence still considerable appeal in this original approach. In this paper we exhibit a modified version of Reid's method that allows DRC branchings to be queried for the desired invariance properties in an entirely algorithmic way, using only differential reduction and completion. This enables us to develop revised branching strategies that can for instance choose the smallest *invariant* pivot (when one is available), and hence give better structured symmetry classifications from DRC. Our method does *not* require knowledge of the equivalence group, nor even of its vector fields: instead it works at the level of determining equations.

2 Mathematical background

2.1 Symmetries of differential equations

Consider a system of s differential equations (DEs) of order $k \geq 1$

$$f^l(x, u, u^{(1)}, \ldots, u^{(k)}) = 0, \quad l = 1, \ldots, s, \tag{1}$$

where $x = (x^1, \ldots, x^n)$ are n independent variables and $u = (u^1, \ldots, u^m)$ are m dependent variables. The 'jet variables' $u^{(k)}$ represent values of k-th derivatives

$$u^{(k)} = \{u^j_I, 1 \le j \le m, I = [i_1 \cdots i_k], 1 \le i_a \le n\} \tag{2}$$

with I an unordered k-tuple. We write $|I| = k$ for the order of I. Let

$$E = \{f^1, \ldots, f^s\} \tag{3}$$

denote the left hand sides of the DEs. We distinguish carefully between E (a set of polynomials in the jet variables) and the *jet variety*

$$\mathcal{V}(E) = \{(x, u, \ldots, u^{(k)}) \in J^k \mid f^l(x, u, \ldots, u^{(k)}) = 0\},$$

which is a set of points in jet space. We shall assume that E contains no 0-th order equations (algebraic equations $f(x, u) = 0$).

The problem of finding sufficient conditions for point symmetries of E is a standard one [1, Sect. 4.3.1], [2, Sect. 2.3], [3, Sect. 5.3]. The starting point is a formal operator on base (x, u)

$$\mathbf{X} = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i} + \sum_{j=1}^{m} \eta^j \frac{\partial}{\partial u^j} \tag{4}$$

(with ξ^i, η^j depending on (x, u)), with its formal prolongation to jet variables

$$\mathbf{X}^{(k)} = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i} + \sum_{j=1}^{m} \eta^j \frac{\partial}{\partial u^j} + \sum_{1 \le |I| \le k} \eta^j_{(I)} \frac{\partial}{\partial u^j_I}, \tag{5}$$

where at this stage all ξ^i, η^j, $\eta^j_{(I)}$ are just symbols. The symmetry procedure is as follows:

1. Apply the operator $\mathbf{X}^{(k)}$ to DEs E to form
$$\mathbf{X}^{(k)} E = \{\mathbf{X}^{(k)} f^1, \ldots, \mathbf{X}^{(k)} f^s\} \tag{6}$$
2. Reduce the expressions $\mathbf{X}^{(k)} E$ modulo the prolongation relations
$$\mathrm{pr} = \left\{ \eta^j_{(Ii)} - D_{x^i} \eta^j_{(I)} + \sum_{l=1}^{n} u^j_{Il} D_{x^i} \xi^l, \quad 0 \le |I| < k \right\}, \tag{7}$$
where D_{x^i} is a total derivative operator and for order 0, $\eta^j_{()}$ is identified with η^j. This reduction amounts to substituting out the prolongation components $\eta^j_{(I)}$ in terms of derivatives of ξ^i, η^j. The resulting set of expressions will be denoted by $\mathbf{X}^{(k)}_{\mathrm{pr}} E$; each is linear homogeneous in ξ^i, η^j and their derivatives, with coefficients that are polynomial in the jet variables $(x, u, \ldots, u^{(k)})$.
3. Restrict $\mathbf{X}^{(k)}_{\mathrm{pr}} E$ to the variety $\mathcal{V}(E)$: we shall denote this by $\mathbf{X}^{(k)}_{\mathrm{pr}} E \,\big|_{\mathcal{V}(E)}$. This is usually achieved by isolating derivatives from each equation of E and substituting them out.

Steps 1 and 2 may be done in either order, in the sense that operator $\mathbf{X}^{(k)}$ (5) can be reduced mod pr (7) before being applied to E.

Depending on one's purpose, one may use the expression $\mathbf{X}^{(k)}_{\mathrm{pr}} E \,\big|_{\mathcal{V}(E)}$ to answer either of two questions:

Construction The question of finding all point symmetries of E is answered by setting $\mathbf{X}^{(k)}_{\mathrm{pr}} E \,\big|_{\mathcal{V}(E)}$ to 0 and decomposing by powers of the jet variables u^j_I of order $|I| \ge 1$ to obtain a set of linear homogeneous partial differential equations for the infinitesimals ξ, η, the *symmetry determining system* $\mathcal{S}$. Solving $\mathcal{S}$ for ξ, η then gives the symmetry operators.

Decision This is the question whether a vector field $\mathbf{X}$ is a symmetry of E. For this one simply checks whether $\mathbf{X}^{(k)}_{\mathrm{pr}} E \,\big|_{\mathcal{V}(E)}$ is 0.

We shall need a variant of the decision problem for which Steps 2 and 3 above must be recast.

For the symmetry procedure to be algorithmic, it must be specified how step 3. is to be done. This will require some assumptions on E to ensure it satisfies a constant rank condition [2, Sect. 2.3]. The reason for our stipulation that E contain no 0-th order equations is that the invariance condition would then no longer give determining equations as PDEs, and standard methods are blocked.

2.1.1 Example 1 Consider 'the' nonlinear heat equation

$$u_t - (K(u)u_x)_x = 0, \tag{8}$$

which is really a class of PDEs parameterised by $K(u) \neq 0$, and let the symmetry operator be

$$\mathbf{X} = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}, \tag{9}$$

where ξ, τ, η depend on (x, t, u). The above method gives symmetry system $\mathcal{S}$ [1,3]

$$\begin{aligned} &\xi_u = \tau_u = \tau_x = 0, \quad K\eta_{xx} - \eta_t = 0, \quad 2\xi_x - \tau_t - \frac{K_u}{K}\eta = 0, \\ &2\eta_{xu} - \xi_{xx} + 2\frac{K_u}{K}\eta_x - \frac{1}{K}\xi_t = 0, \quad \eta_{uu} + \frac{K_u}{K}\eta_u + \frac{KK_{uu} - K_u^2}{K^2}\eta = 0. \end{aligned} \tag{10}$$

2.2 Algorithmic symmetry classification

Modern symmetry packages [5,27,28] deal with symmetry determining equations $\mathcal{S}$ in three stages: *construct* $\mathcal{S}$ (see Sect. 2.1); *reduce and complete* $\mathcal{S}$ using a differential reduction and completion (DRC) algorithm; *solve* $\mathcal{S}$. Solving is not algorithmic, but the first two stages are, and it is of considerable interest to extract as much information as possible about symmetries without solving $\mathcal{S}$. A DRC algorithm casts a DE system into a form where a local existence-uniqueness theorem can give a clear correspondence between the DEs and their solutions; reaching this form uses only differentiation and polynomial algebra. Properties such as dimension of solution space [16] and (in the case of symmetry determining equations) structure constants of the Lie symmetry algebra [29] can then be found without solving the DEs.

There are several practical variants of DRC algorithms. Riquier–Janet theory [28,30], [31, Sect. 2.2] applies to linear PDE, such as symmetry determining equations. Unfortunately, determining systems for symmetry classification contain not just the infinitesimals ξ, η (which occur linearly), but a second category of dependent variable, namely the arbitrary elements a. For example in system (10), diffusivity K occurs nonlinearly. Hence a fully nonlinear DRC theory is needed. Such theories include the *rif* (reduced involutive form) of Reid and Wittkopf [16,22] and the Rosenfeld–Gröbner algorithm of Boulier et al. [17].

The structure that is desired for the determining equations $\mathcal{S}$ after DRC is that $\mathcal{S}$ contain a subsystem for the arbitrary elements a, with the rest of the equations being linear in the infinitesimals ξ, η. We define the following notation:

Definition 2 Let Θ be a subset of the set of dependent variables in a DEs system, and let $\bar{\Theta}$ be its complement. We write $\Theta \ll \bar{\Theta}$ if every derivative of each $\theta \in \Theta$ is ranked lower than every derivative of each $\bar{\theta} \in \bar{\Theta}$.

Such an elimination ranking has the property that it forces out a subsystem for the lower ranked variables Θ. Thus in $\mathcal{S}$ a ranking for which $\{a\} \ll \{\xi, \eta\}$ will ensure that $\mathcal{S}$ has the desired structure. Examples applying *rif* to symmetry classification are given in [16,22].

2.2.1 Example 3 Consider the symmetry determining the equations of (10) for the nonlinear heat equation, and choose a ranking in which $\{K\} \ll \{\xi, \tau, \eta\}$. For a wide variety of rankings, *rif* case splits on the three expressions

$$p_1 = K_u, \quad p_2 = 4KK_{uu} - 7K_u^2, \quad p_3 = KK_uK_{uuu} + K_u^2K_{uu} - 2KK_{uu}^2. \tag{11}$$

One leaf of the resulting classification tree is for instance

$$K_u \neq 0, \quad 4KK_{uu} - 7K_u^2 = 0, \\ \xi_t = \xi_u = \tau_x = \tau_u = \eta_t = \eta_{xx} = 0, \tag{12a}$$

$$\xi_{xx} = \frac{1}{2}\frac{K_u}{K}\eta_x, \quad \tau_t = 2\xi_x - \frac{K_u}{K}\eta, \quad \eta_u = -\frac{3}{4}\frac{K_u}{K}\eta. \tag{12b}$$

Classifying system (12a) shows this is a two-parameter family of $K(u)$, while (12b) shows that a 5-dimensional symmetry algebra is admitted by (8) for these K. Equation 12a can be solved to find $K(u) = K_0(u-b)^{-4/3}$ (with $K_0 \neq 0$) but *this is not required to achieve classification*.

The Rosenfeld–Gröbner algorithm [17] may instead be used, but the ideas and results are similar. The advantage of such methods is that they cleanly separate *symmetry classification* (achieved algorithmically using DRC) from *finding symmetries* (which requires solving PDE).

2.3 Equivalence Group

A class of differential equations has an associated collection of transformations which map DEs to DEs in the class C. The *equivalence group* [3, Sect. 6.4], [32], [33, Chap. 2] (or 'structure invariance group' [31, Chap. 4]) is the set of those transformations which act as point transformations on the space (x, u, a) (where a are coordinates representing the arbitrary elements) and which map every member of C to a member of C. This group is useful for parameter removal: the *Handbook* [7, Part B] uses it systematically in presenting results of symmetry classifications.

2.3.1 Example 4 For the class of nonlinear heat equation (8), the equivalence group acts as point transformations on (x, t, u, K); the corresponding operators are

$$\mathbf{Y} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + \kappa\frac{\partial}{\partial K}, \tag{13}$$

where ξ, τ, η depend on (x, t, u), while κ depends on (x, t, u, K). The equivalence group in this case represents degrees of freedom relating to choice of origin and of units for length, time and temperature. It establishes the equivalence

$$K(u) \leftrightarrow \frac{1}{K_0}K(au+b), \quad K_0 \neq 0, a \neq 0,$$

where K_0, a, b are arbitrary constants.

Following [3, Sect. 6], [32], the method for finding equivalence operators is analogous to finding symmetries. Starting with the operator

$$\mathbf{Y} = \sum_{i=1}^{n}\xi^i\frac{\partial}{\partial x^i} + \sum_{j=1}^{m}\eta^j\frac{\partial}{\partial u^j} + \sum_{l=1}^{p}\alpha^l\frac{\partial}{\partial a^l}, \tag{14}$$

where ξ^i, η^j depend on (x, u) and α^l depend on (x, u, a), one constructs a system $\mathcal{Q}$ of determining equations for the infinitesimals ξ, η, α. We shall call $\mathcal{Q}$ the *equivalence system*.

2.3.2 Example 5 For the nonlinear heat equation (8), the equivalence operator $\mathbf{Y}$ (13) is found to have infinitesimals ξ depending on x; τ on t, η on u, and κ on K, and obeying the equivalence system $\mathcal{Q}$

$$\xi_x = \frac{1}{2}\tau_t + \frac{1}{2}\frac{1}{K}\kappa, \quad \tau_{tt} = 0, \quad \eta_{uu} = 0, \quad \kappa_K = \frac{1}{K}\kappa. \tag{15}$$

The equivalence group Γ plays a central role in clarifying and simplifying symmetry classification. Unfortunately, the *rif* DRC algorithm applied to symmetry classification (Sect. 2.2) knows nothing of Γ. An undesirable feature in this case is that *rif* may choose branchings that are not Γ-invariant; in this case two equations known to be equivalent appear in different branches of the classification.

Methods such as described by [10,25,26] can perform symmetry classification using DRC with respect to a noncommuting basis of differential operators. The symmetry system is written in invariant form by choosing the operators to be invariant under the equivalence group Γ. To compute invariants, one assumes there is an explicit parametrisation of Γ. In this paper we take a different approach. We not only work infinitesimally, we do not even assume that the equivalence operators (14) are known. Instead, our method uses the *equivalence system* $\mathcal{Q}$ to assist in performing DRC on the *symmetry system* $\mathcal{S}$. This has the advantage of being completely algorithmic and requiring no explicit solution of DEs, and no explicit parametrisation of groups.

3 Symmetry classification with invariance

Because branchings in a symmetry classification should be Γ-invariant, this implies that all transformations in the equivalence group Γ should be symmetries of the branching conditions. Our goal is to show how to check whether a given branching condition is Γ-invariant *directly from the determining equations for* Γ. By doing this we can guide the *rif* algorithm towards more desirable branchings. To achieve this goal careful reassessment of the standard symmetry method described in Sect. 2.1 is needed.

3.1 Symmetry condition

Consider step 3 of the symmetry procedure described in Sect. 2.1, namely "restricting to the variety $\mathcal{V}(E)$". For DE systems arising in physics or engineering it is usually adequate to do this by isolating a derivative from each equation in E and substituting it out. Experience with the Maple `rifsimp` DRC package showed this to be inadequate for our purposes. Case splitting conditions arising during a symmetry classification can be highly nonlinear (e.g. $y(y''')^2 + y'y''y''' + (y'')^3 = 0$). In this case 'isolating the leading derivative y'''' is not algebraically achievable. And consider the following example, where the equation is 'leading linear', yet isolating the leading derivative gives incorrect results.

3.1.1 Example 6 Consider the DE

$$E = 0, \quad \text{where } E = \{y_{xx}(z_{xx} + z)\} \tag{16}$$

with solutions

$$\{y = ax + b, z = g(x)\} \cup \{y = h(x), z = c\sin x + d\cos x\}. \tag{17}$$

Seek a symmetry vector field $\mathbf{X} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}$, where to cut down expression size we stipulate that ξ depends on x, η on (x, y) and ζ on (x, z). Applying the method of Sect. 2.1 gives

$$\mathbf{X}^{(2)}E = (z_{xx} + z)\eta_{(xx)} + y_{xx}\zeta_{(xx)} + y_{xx}\zeta \tag{18}$$

and after reducing mod prolongations (7):

$$\begin{aligned}\mathbf{X}^{(2)}_{\text{pr}}E = {} & (z_{xx} + z)\left(\eta_{xx} + 2\,y_x\eta_{xy} - y_x\xi_{xx} + y_{xx}\eta_y - 2\,y_{xx}\xi_x + y_x^2\eta_{yy}\right)\\ & + y_{xx}\left(\zeta_{xx} + 2\,z_x\zeta_{xz} - z_x\xi_{xx} + z_x^2\zeta_{zz} + z_{xx}\zeta_z - 2\,z_{xx}\xi_x\right) + y_{xx}\zeta.\end{aligned} \tag{19}$$

Attempting to restrict to the variety $\mathcal{V}(E)$ by 'substituting out the leading derivative' (e.g. set $y_{xx} = 0$) gives determining equations whose solutions incorrectly include operators like $x\frac{\partial}{\partial x}$ which are not symmetries of E (16). For instance the solution (17) $y = \mathrm{e}^x, z = \cos x$ is mapped by the scaling group generated by $x\frac{\partial}{\partial x}$ to

$y = \mathrm{e}^{ax}, z = \cos ax$, which is not a solution for $a \neq \pm 1$. The fact that different 'symmetries' are found if z_{xx} is chosen as the leading derivative of E also indicates something seriously amiss. The example may appear artificial, but case splitting conditions of this kind do arise when using `rifsimp`. The cause of the difficulty is that 'substituting out the leading derivative' restricts to an incorrect variety.

The question of whether an expression vanishes on an algebraic variety $\mathcal{V}(E)$ is a staple of commutative algebra (see e.g. [34, Chap. 4]). Let the ideal generated by E be denoted by $\mathcal{I}(E)$. If $\mathcal{I}(E)$ is radical, the question is answered algorithmically by reduction modulo a Gröbner basis for $\mathcal{I}(E)$. If $\mathcal{I}(E)$ is not radical, a basis for the radical of $\mathcal{I}(E)$ can still be found algorithmically [35, Sect. 4.5]. (Methods using the idea of characteristic sets (e.g. [17]) can do more in this direction.)

Assuming that $\mathcal{I}(E)$ is radical, step 3 of the symmetry method of Sect. 2.1 can therefore be replaced by

3.′ Restrict $\mathbf{X}^{(k)}_{\mathrm{pr}} E$ to the variety $\mathcal{V}(E)$ by reducing all the polynomial coefficients modulo a Gröbner basis $\mathcal{I}(E)$ with respect to a ranking (Definition 2) in which $\{u\} \ll \{u^{(1)}, \ldots u^{(k)}\}$.

The reason for the restriction on ranking is that ξ, η may depend on (x, u) but not on derivatives. Conveniently, a Gröbner basis in such a ranking will exhibit a basis for any algebraic equations $f(x, u)$ in $\mathcal{I}(E)$, enabling algorithmic checking of the assumption made in Sect. 2.1 that $\mathcal{I}(E)$ contain no algebraic equations.

3.1.2 Example 6 (cont.) Note that DE E (16) generates a radical ideal. Applying step 3.′ amounts to taking remainder on division by $y_{xx}(z_{xx} + z)$, which gives

$$\begin{aligned}\mathbf{X}^{(2)}_{\mathrm{pr}}\big|_{\mathcal{V}(E)} &= z\eta_{xx} + y_x(2\,z\eta_{xy} - z\xi_{xx}) + y_x^2 z\eta_{yy} + y_{xx}(\zeta_{xx} + \zeta + 2z\,\xi_x - z\zeta_z)\\ &\quad + z_{xx}\eta_{xx} + z_x y_{xx}(2\,\zeta_{xz} - \xi_{xx}) + y_x z_{xx}(2\,\eta_{xy} - \xi_{xx}) + z_x^2 y_{xx}\zeta_{zz} + y_x^2 z_{xx}\eta_{yy}.\end{aligned}$$

Decomposing by powers of derivatives and solving the resulting determining equations gives the correct list of symmetries of E, (16)

$$\frac{\partial}{\partial x},\quad \frac{\partial}{\partial y},\quad x\frac{\partial}{\partial y},\quad y\frac{\partial}{\partial y},\quad \sin x\frac{\partial}{\partial z},\quad \cos x\frac{\partial}{\partial z},\quad z\frac{\partial}{\partial z},$$

all of which map solutions (17) to solutions.

3.2 Invariance checking

Before proceeding to our main purpose of symmetry classification, we first address a subsidiary question. For a differential equation E, we *may* be interested in finding all point symmetries of E, but there are circumstances where we wish to consider only those point symmetries from a specified (pseudo-)group G. The infinitesimals of G are specified by a system of determining equations $\mathcal{D}$. (In the sequel, $\mathcal{D}$ will be the equivalence system.)

3.2.1 Example 7 Consider the DE $y_{xx} = 0$ (so that $E = \{y_{xx}\}$) and suppose that rather than all point symmetries, we seek only conformal symmetries of E. Thus the operator $\mathbf{X} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}$ is to have infinitesimals ξ, η (depending on (x, y)) that satisfy the Cauchy–Riemann equations

$$\mathcal{D} = \{\eta_y = \xi_x,\quad \xi_y = -\eta_x\}. \tag{20}$$

This example immediately suggests a solution: set up the point symmetry system for E, then append relations $\mathcal{D}$ to it. However, this is not well-suited to our purposes in Sect. 3.3, and we instead give an adjusted procedure that avoids construction of the point symmetry system.

Suppose that symmetry vector fields are being sought in a group of point transformations specified by determining equations $\mathcal{D}$. We seek to adjust the symmetry method of Sect. 2.1 so that at step 2 we take account not only of the prolongation relations pr (7) but also the specified determining system $\mathcal{D}$. Both pr and $\mathcal{D}$ are linear homogeneous partial differential equations for the components of the prolonged vector field. Specify a derivative ranking (Definition 2) such that

$$\{\xi, \eta\} \ll \{\eta^{(1)}\} \ll \cdots \ll \{\eta^{(k)}\} \tag{21}$$

(where again the components of $\xi, \eta, \ldots, \eta^{(k)}$ are symbols). When a DRC algorithm is applied to the combined system $\mathrm{pr} \cup \mathcal{D}$, such a ranking ensures

i. that determining system $\mathcal{D}$ (which involves only ξ, η and derivatives) still appears as a subsystem.
ii. that the prolongation relations pr (7) act as substitution rules that eliminate the prolongation components $\eta^{(1)}$, …, $\eta^{(k)}$ in favour of expressions in terms of ξ, η and derivatives.

3.2.2 Example 7 (cont.) Choose a ranking for which

$$\{\xi, \eta\} \ll \eta_{(x)} \ll \eta_{(xx)}$$

and apply the *rif* algorithm [16] to $\mathrm{pr} \cup \mathcal{D}$ (7),(20)

$$\{\eta_y - \xi_x, \quad \xi_y + \eta_x, \quad \eta_{(x)} - \eta_x - y_x\eta_y + y_x\xi_x + y_x^2\xi_y,$$
$$\eta_{(xx)} - \eta_{(x)\,x} - y_x\eta_{(x)\,y} - y_{xx}\eta_{(x)\,y_x} + y_{xx}\xi_x + y_x y_{xx}\xi_y\},$$

where we break ties by ranking x-derivatives lower than y-derivatives. The *rif*-form redpr of the combined system is

$$\mathrm{redpr} = \{\eta_y - \xi_x, \quad \xi_y + \eta_x, \quad \eta_{(x)} - (1 + y_x^2)\eta_x, \eta_{(xx)} - (1 + y_x^2)(\eta_{xx} + y_x\xi_{xx}) - 3y_x y_{xx}\eta_x + y_{xx}\xi_x\}. \tag{22}$$

The last two expressions in particular define prolongation formulas adapted to conformal vector fields. With this, step 2 of the symmetry procedure of Sect. 2.1 is adjusted to the following. Given system of DEs E and determining equations $\mathcal{D}$ for the infinitesimals of some group of point transformations, precompute the *rif*-form redpr of $\mathrm{pr} \cup \mathcal{D}$ as described above. Let the *rif*-form of $\mathcal{D}$ be denoted by $\mathcal{D}_{\mathrm{red}}$.

2′. Reduce the expressions $\mathbf{X}^{(k)}E$ modulo the *rif*-form redpr of $\mathrm{pr} \cup \mathcal{D}$. This reduction amounts to substituting out the 0-th order principal derivatives from $\mathcal{D}_{\mathrm{red}}$ in favour of parametric derivatives. (This includes substituting out the prolongation components $\eta^j_{(I)}$.) The resulting set of expressions will be denoted by $\mathbf{X}^{(k)}_{\mathrm{redpr}}E$; each is linear homogeneous in the parametric derivatives of ξ^i, η^j, with coefficients that are polynomial in the jet variables $(x, u, \ldots, u^{(k)})$.

The expressions $\mathbf{X}^{(k)}_{\mathrm{redpr}}E\,\big|_{\mathcal{V}(E)}$ that result from steps 1, 2′, 3′ can be used similarly to before:

Construction To construct all symmetries of E from the group defined by $\mathcal{D}$, set $\mathbf{X}^{(k)}_{\mathrm{redpr}}E\,\big|_{\mathcal{V}(E)}$ to 0, decompose by powers of the jet variables u^j_I of order $|I| \geq 1$ to obtain additional determining equations. Append these to $\mathcal{D}$ to get the symmetry system $\mathcal{S}$ of E.

Decision The question whether all solutions of determining system $\mathcal{D}$ are symmetries of E can be answered by checking whether $\mathbf{X}^{(k)}\mathrm{redpr}\,E\,\big|_{\mathcal{V}(E)}$ is 0.

3.2.3 Example 7 (cont.) Write the formal prolonged operator

$$\mathbf{X}^{(2)} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \eta_{(x)}\frac{\partial}{\partial y_x} + \eta_{(xx)}\frac{\partial}{\partial y_{xx}}.$$

Applying the modified symmetry procedure, we have

1. $\mathbf{X}^{(2)}E = \{\mathbf{X}^{(2)}y_{xx}\} = \{\eta_{(xx)}\}$;

2′. Reduce $\mathbf{X}^{(2)}E$ mod redpr (22) to get

$$\mathbf{X}^{(2)}_{\text{redpr}}E = \{(1+y_x^2)(\eta_{xx}+y_x\xi_{xx})+3y_xy_{xx}\eta_x - y_{xx}\xi_x\};$$

3′. Reduce mod the Gröbner basis $E=\{y_{xx}\}$ to get

$$\mathbf{X}^{(2)}_{\text{redpr}}E\,\big|_{\mathcal{V}(E)} = \{(1+y_x^2)(\eta_{xx}+y_x\xi_{xx})\}.$$

The fact that this expression $\mathbf{X}^{(2)}_{\text{redpr}}E\,\big|_{\mathcal{V}(E)}$ is nonzero indicates that not all conformal transformations are symmetries of $y_{xx}=0$. Construction of the conformal symmetries of $y_{xx}=0$ is achieved by breaking down by powers of y_x to get $\eta_{xx}=0$, $\xi_{xx}=0$. These, along with the Cauchy–Riemann equation (20) constitute the determining equations of the conformal symmetries of $y_{xx}=0$. Notice that this procedure allows one to check the invariance of DE system E under group G without knowing G, nor even the vector fields generating G—just the infinitesimal determining equations of G.

For symmetries of a single system of DEs E, the above procedure has only minor advantages over the simplistic one of appending $\mathcal{D}$ to the point symmetry system of E. However, when multiple systems $E_1, E_2, \ldots$ are being analysed for invariance under the same group G, there is a major gain, because the reduced prolongation relations are precomputed, and can be reused on each system. For each system E_l, step 1 is just differentiation of E_l, while the 'reduction' of step 2′ amounts to simple substitution from the reduced prolongation relations. Hence it is only step 3 (reduction mod E) that is potentially expensive, especially when E is complicated. For our application to symmetry classification using *rif* even this expense is modest: E_l is a single equation so 'reduction mod E' is computing remainder on division by a single polynomial.

3.3 Classification with invariance checking

All the pieces are now in place to test branchings in a symmetry classification for invariance. Starting with a system E of differential equations involving arbitrary elements a, derive determining equations $\mathcal{Q}$ for the equivalence transformations. Then execute a differential reduction and completion (DRC) algorithm on the symmetry system $\mathcal{S}$.

In the simplest version, branchings of the DRC algorithm can be tested after the fact for invariance under $\mathcal{Q}$. We sweep through the tree, testing the many case splitting conditions for invariance with respect to the same $\mathcal{Q}$.This is where the comments about efficiency at the end of Sect. 3.2 become significant.

3.3.1 Example 8 Consider the equivalence group (Example 5) of the nonlinear heat equation (8) as projected onto action on u, K. The operator is $\mathbf{Y}=\eta\frac{\partial}{\partial u}+\kappa\frac{\partial}{\partial K}$ where η depends on u and κ on K; the determining equations are

$$\eta_{uu}=0, \quad \kappa_K=\frac{1}{K}\kappa. \tag{23}$$

This system is in *rif* form. Applying the theory of Sect. 3.2, we find reduced prolongations mod $\mathcal{Q}$:

$$\left\{\kappa_{(u)}-\frac{K_u}{K}\kappa+K_u\eta_u, \quad \kappa_{(uu)}-\frac{K_{uu}}{K}\kappa+2K_{uu}\eta_u\right\}. \tag{24}$$

The *rif* algorithm splits the symmetry system on the three expressions p_1, p_2, p_3 of (11). For instance consider $p_2=4KK_{uu}-7K_u^2$: we ask if this is invariant under the equivalence group action defined by $\mathcal{Q}$ (23). Begin with the formal prolonged operator

$$\mathbf{Y}^{(2)}=\eta\frac{\partial}{\partial u}+\kappa\frac{\partial}{\partial K}+\kappa_{(u)}\frac{\partial}{\partial K_u}+\kappa_{(uu)}\frac{\partial}{\partial K_{uu}}$$

and apply the procedure of Sect. 3.2:

1. Apply $\mathbf{Y}^{(2)}$ to p_2:

$$\mathbf{Y}^{(2)}p_2=4K\kappa_{(uu)}-14K_u\kappa_{(u)}+4K_{uu}\kappa;$$

2′. Reduce mod the reduced prolongation relations (24):

$$\mathbf{Y}^{(2)}_{\text{redpr}} p_2 = \left(-\frac{2}{K}\kappa + \eta_u\right)(4K K_{uu} - 7K_u^2);$$

3′. Reduce mod p_2, which gives 0.

Hence we conclude that classifying condition $p_2 = 0$ (11) is invariant under the equivalence group action. In fact, all three splittings, on p_1, p_2, p_3 (11) test as invariant under the equivalence group defined by $\mathcal{Q}$ (23). For the nonlinear heat equation, this *post hoc* tagging of case splits gives us confidence that the *rif* classification tree is of good quality. Another example, where *post hoc* labelling points up a problem in a *rif* classification is given in Sect. 3.5 (Fig. 1).

Invariance checking can instead be used to *guide* `rifsimp` as it performs symmetry classification. When `rifsimp` has a choice of expressions on which it might branch, typically some heuristic is applied: it might choose the condition of least size, lowest order, etc. The theory of Sect. 3.2 offers another possibility: namely to prefer the *invariant* candidate of least size, lowest order, etc. We give an example in Sect. 3.5 (Fig. 2).

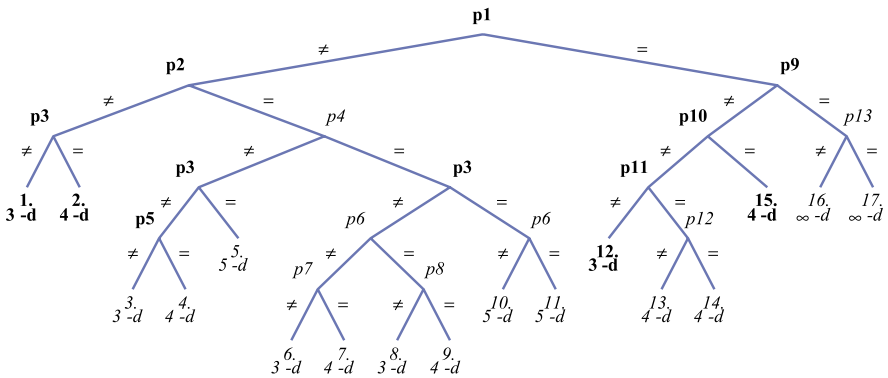

Fig. 1 Classification tree for 1+1 Richards equation in potential form using vanilla `rifsimp` with ranking $\{B\} \ll \{K\} \ll \{\xi, \tau, \eta, \phi\}$. Pivots $p1, \ldots, p13$ are case splitting conditions. Invariant case splits are shown in upright **bold**, non-invariant in *italic*. The dimension of the symmetry group is also shown at each leaf

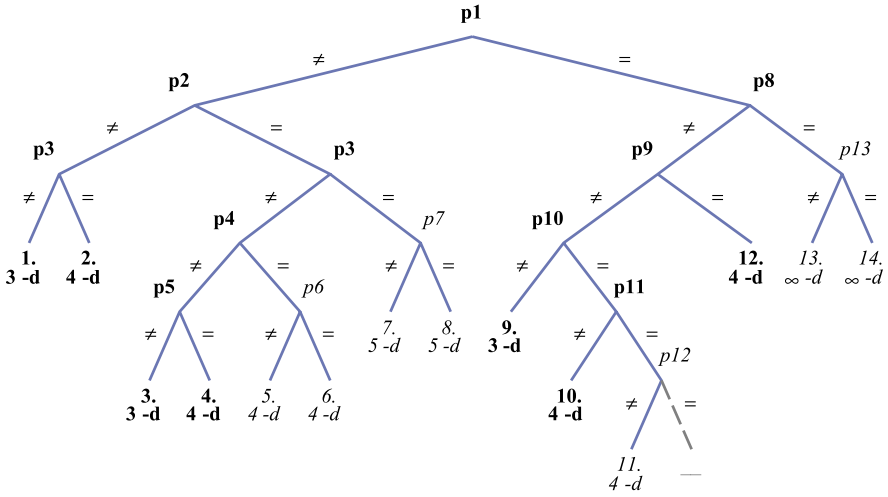

Fig. 2 Classification tree for 1+1 Richards equation in potential form, using `SymmetricRifsimp` to guide pivot selection. The pivot numbering is not the same as in Fig. 1. Note that there are now three fewer cases. The *dashed branch* is inconsistent so is not counted as a case

3.4 Implementation

Many symmetry packages are available for computer-algebra systems. We chose to implement the invariant symmetry classification method in Maple, utilising a suite of symmetry analysis functions added to the `PDEtools` package at Version 11, plus the `rifsimp` command, which implements Reid and Wittkopf's *rif* algorithm [16,22]. We have written a `SymmetryClassification` package, containing about 3000 lines of code, and which has the following features:

- The differential equations E, their symmetry system $\mathcal{S}$ and equivalence system $\mathcal{Q}$ are stored in a Maple `module` structure, since ultimately they are all attributes of E.
- Our package utilises the Maple `PDEtools:-DeterminingPDE` function to find $\mathcal{S}$ and adds another function for finding $\mathcal{Q}$.
- There is a function to tag a `rifsimp` symmetry classification *post hoc* with invariance information. Additionally, a modified version of `rifsimp` (tentatively called `SymmetricRifsimp`) is provided with an additional `invariant` pivot selection strategy that case splits on the smallest *invariant* pivot.
- There is a function analogous to `DEtools[caseplot]` for displaying classification trees tagged with invariance information.

The package is available for download at http://ise.canberra.edu.au/ianlisle.

3.5 Example: 1+1 Richards

We now apply the invariance checking method to a more substantial example using the `SymmetryClassification` package. Consider the 1+1 Richards equation in potential form

$$v_x = u, \quad v_t = B(u)u_x - K(u), \quad B(u) \neq 0,$$

where u, v are the functions of (x, t), and $B(u)$, $K(u)$ are the arbitrary elements. This system was classified by Lisle and Reid [10] by a different method. The equivalence vector field is

$$\mathbf{Y} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + \phi\frac{\partial}{\partial u} + \beta\frac{\partial}{\partial B} + \kappa\frac{\partial}{\partial K}$$

with ξ, τ, η, ϕ depending on (x, t, u, v) and β, κ on (x, t, u, v, B, K).

The equivalence system $\mathcal{Q}$ as projected onto (u, B, K) space is

$$\begin{aligned} &\eta_{uuu} = 0, \quad \beta_B = \tfrac{1}{B}\beta, \qquad \beta_u = -B\eta_{uu}, \\ &\kappa_{KK} = 0, \quad \kappa_{uK} = \tfrac{1}{2}\eta_{uu}, \quad \kappa_{uu} = 0, \end{aligned} \tag{25}$$

where η depends on u; β on (u, B); and κ on (u, K). The symmetry system $\mathcal{S}$ is [10]:

$$\begin{aligned} &\tau_u = 0, \quad \tau_v = 0, \quad \tau_x = 0, \quad \xi_u = 0, \quad \phi_u = 0, \\ &B(\phi_v + \xi_x - \tau_t - \eta_u) - B'\eta = 0, \\ &\phi_x + u\phi_v - u(\xi_x + u\xi_v) - \eta = 0, \\ &\phi_t - K\phi_v - u(\xi_t - K\xi_v) + K\tau_t + K'\eta - B(\eta_x + u\eta_v) = 0. \end{aligned} \tag{26}$$

Note that the η's in (25,26) are distinct. From $\mathcal{Q}$ (25), we find there is a 7-parameter equivalence group action on (u, B, K) [10]. Applying `rifsimp` to $\mathcal{S}$ with ranking $\{B\} \ll \{K\} \ll \{\xi, \tau, \eta, \phi\}$ to classify symmetries gives a classification tree as shown in Fig. 1, tagged *post hoc* with invariance information. The pivots are fairly complex, having up to 10 terms with derivatives up to 5th order. Despite this the labelling sweep takes only about 0.35 sec.

In Fig. 1 note that `rifsimp` has early on chosen a non-invariant case split (on $p4$). The left and right subtrees below the $p4$ split are very similar, having cases with 3-, 4- and 5-dimensional symmetry groups. This is presumably because these two subtrees are in fact connected by an equivalence transformation, leading to a repetition

in structure. Similar comments apply to the $p6$ split. These bad splittings are because vanilla `rifsimp` takes no account of the equivalence group.

We now compare these results with those obtained when `SymmetricRifsimp` is used to guide case splits according to their invariance properties. With the same ranking, the revised classification tree is shown in Fig. 2.

The invariant `SymmetricRifsimp` method only splits on a non-invariant pivot when no invariant pivot is available. As a result it has rejected pivot $p4$ chosen by vanilla `rifsimp` in Fig. 1 and has deferred non-invariant pivots almost down to the leaves of the tree. The repeated subtrees disappear and the number of cases drops by 3. The performance penalty for the improved tree is modest: Maple timings gave 6.5 s for vanilla `rifsimp` and 7.5 s for `SymmetricRifsimp`.

3.6 Completion of classification

The case splittings and symmetry group dimensions provided by a DRC method such as *rif* are only the first step towards a complete group classification. Ultimately we wish to solve the classifying equations to find explicit forms of the arbitrary elements, and to solve the other symmetry determining equations to find infinitesimals. This part of the process is not strictly algorithmic, but it is not as daunting as may first appear. An invariant case splitting is rich in symmetry because every equivalence operator is a symmetry.

As an example, consider again the 1+1 Richards equation of Sect. 3.5, but this time with a different ranking. We still take $\{B\} \ll \{K\} \ll \{\tau, \xi, \phi, \eta\}$, with ties broken by order of derivative; but now we order lexicographically by independent variable $u < v < x < t$ then lexicographically by dependent variable $\eta < \phi < \xi < \tau$. Using the invariant `rifsimp` algorithm of Sect. 3.3 gives a tree with 11 leaves and 8 pivots:

$$
\begin{aligned}
p_1 &= 2BB'' - 3B'^2,\\
p_2 &= B(2BB'' - 3B'^2)B^{(4)} - 3B^2B'''^2 + 16BB'B''B''' - 6B'^3B''' - 6B''^2(2BB'' - B'^2),\\
p_3 &= K'',\\
p_4 &= -2B(2BB'' - 3B'^2)(K''K^{(4)} - K'''^2),\\
&\quad + (2B^2B''' + 6B'^3 - 8BB'B'')K''K''' + (-3BB'B''' + 6BB''^2 - 3B'^2B'')K''^2,\\
p_5 &= 2BK''' - 3B'K'',\\
p_6 &= -B^2K^{(4)} + 4BB'K''' - 3B'^2K'',\\
p_7 &= B(2BK''' - 3B'K'')K''K^{(5)} - 4B^2K''K^{(4)^2} + 2B^2K'''^2K^{(4)},\\
&\quad + 11BB'K''K'''K^{(4)} + 3B'^2K''^2K^{(4)} - 8BB'K'''^3 - 4B'^2K''K'''^2,\\
p_8 &= B'.
\end{aligned}
$$

Pivots p_1–p_7 test as invariant, while p_8 is not. The classifying systems are:

$$
\begin{aligned}
&\text{Leaf 1:} && p_1 \neq 0, && p_2 \neq 0, && p_3 \neq 0, && \\
&\text{Leaf 2:} && p_1 \neq 0, && p_2 \neq 0, && p_3 = 0, && \\
&\text{Leaf 3:} && p_1 \neq 0, && p_2 = 0, && p_3 \neq 0, && p_4 \neq 0,\\
&\text{Leaf 4:} && p_1 \neq 0, && p_2 = 0, && p_3 \neq 0, && p_4 = 0,\\
&\text{Leaf 5:} && p_1 \neq 0, && p_2 = 0, && p_3 = 0, && \\
&\text{Leaf 6:} && p_1 = 0, && p_5 \neq 0, && p_6 \neq 0, && p_7 \neq 0,\\
&\text{Leaf 7:} && p_1 = 0, && p_5 \neq 0, && p_6 \neq 0, && p_7 = 0,\\
&\text{Leaf 8:} && p_1 = 0, && p_5 \neq 0, && p_6 = 0, && p_8 \neq 0,\\
&\text{Leaf 9:} && p_1 = 0, && p_5 \neq 0, && p_6 = 0, && p_8 = 0,\\
&\text{Leaf 10:} && p_1 = 0, && p_5 = 0, && p_8 \neq 0, && \\
&\text{Leaf 11:} && p_1 = 0, && p_5 = 0, && p_8 = 0. &&
\end{aligned}
$$

Several of the splitting equations are very easy to solve; we indicate how to solve the more complex ones. For this it is desirable to have the equivalence group explicitly, found by solving system $\mathcal{Q}$ (25). The action on u is

$$u = \frac{\alpha\tilde{u} + \beta}{\gamma\tilde{u} + \delta}$$

and the effect on $B(u)$, $K(u)$ is

$$\tilde{B}(\tilde{u}) = \frac{\rho}{(\gamma\tilde{u} + \delta)^2} B\left(\frac{\alpha\tilde{u} + \beta}{\gamma\tilde{u} + \delta}\right), \tag{27a}$$

$$\tilde{K}(\tilde{u}) = \lambda(\gamma\tilde{u} + \delta) K\left(\frac{\alpha\tilde{u} + \beta}{\gamma\tilde{u} + \delta}\right) + \mu\tilde{u} + \epsilon, \tag{27b}$$

where $\alpha, \beta, \gamma, \delta, \rho, \lambda, \mu, \epsilon$ are group parameters satisfying $\alpha\delta - \beta\gamma = 1$, $\rho, \lambda \neq 0$.

Consider Leaf 3, where we must solve $p_2 = 0$. From $\mathcal{Q}$ (25), this 4th order equation admits 4 symmetries. Rather than directly substituting differential invariants as in [1], we find first integrals, of which there are 4. Eliminating higher derivatives from these shows that B obeys the first-order equation

$$\frac{B'}{B} = \frac{2c_2 - 2c_3 u}{c_3 u^2 - 2(c_2 + c_4)u + c_1},$$

which is a quadrature. Similarly, for Leaf 4 one can solve $p_4 = 0$ by finding a first integral and eliminating using the above result for B; K satisfies the 3rd order equation

$$\frac{K'''}{K''} = \frac{3(c_2 - c_3 u) + c_5}{c_3 u^2 - 2(c_2 + c_4)u + c_1},$$

which again can be solved by quadratures. For Leaves 6–11, diffusivity B satisfies $p_1 = 0$, so that $B(u) = 1/(au + b)^2$. This is equivalent to $B(u) = 1$ by (27) so for Leaf 7 we can simplify $p_7 = 0$ by substituting $B = 1$. After finding two first integrals, K is found to satisfy the equation

$$\frac{K'''}{K''} = \frac{1}{c_1 u + c_3},$$

which is easily solved. The full symmetry classification of the 1+1 Richards equation is shown in Table 1, where parameters have been removed as much as possible by (27).

4 Discussion

The advantage of DRC methods is that they are completely algorithmic, using only differentiation and polynomial algebra. By working at the level of determining equations our invariance checking is able to retain this desirable property. We have no need to construct groups, or invariants of groups, nor even to construct vector fields by solving determining equations.

Our method does not of its itself deal with the expression swell that can inhibit completion of a classification via DRC, especially when the class of DEs being analysed has many degrees of freedom in its arbitrary elements and its equivalence group. In this case there is little alternative other than to find the action of the equivalence group explicitly and to apply a more geometric approach, such as described in [10,25]. Although somewhat against the spirit of our method, our Maple code does make some concessions in this direction, by allowing the user to define differential invariants of the equivalence group (if known) [10] and to specify these as 'constraints' on the arbitrary elements. This can avert expression swell to a remarkable extent.

The choice of ranking used in applying DRC methods to a symmetry system also has a very strong effect on the quality of the classification produced. Experience with our method indicates that an ill-chosen ranking can lead to intractable expression swell and a poor quality tree, even when `rifsimp` is being guided towards invariant pivots.

Table 1 Symmetry classification (up to real-valued equivalence (27)) for the 1+1 Richards equation in potential form

Case 2 Generic nonlinear diffusion, linear convection		
	$B(u)$ arbitrary, $K(u)=0$	$\mathbf{X}_4 = v\frac{\partial}{\partial v} + x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}$
Case 4 Nonlinear diffusion, nonlinear convection		
a.	$B(u)=u^m$, $K(u)=u^n$ $m\neq 0,-2,\ \ n\neq 0,1$	$\mathbf{X}_4 = (m-n+2)v\frac{\partial}{\partial v} + (m-n+1)x\frac{\partial}{\partial x}$ $+(m-2n+2)t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$
b.	$B(u)=u^m$, $K(u)=u\log u$ $m\neq 0,-2$	$\mathbf{X}_4 = (m+1)v\frac{\partial}{\partial v} + (t+mx)\frac{\partial}{\partial x} + mt\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$
c.	$B(u)=\mathrm{e}^{mu}$, $K(u)=\mathrm{e}^{u}$ $m\neq 0$	$\mathbf{X}_4 = (x+(m-1)v)\frac{\partial}{\partial v} + (m-1)x\frac{\partial}{\partial x}$ $+(m-2)t\frac{\partial}{\partial t}$
d.	$B(u)=\mathrm{e}^{u}$, $K(u)=u^2$	$\mathbf{X}_4 = (v+x)\frac{\partial}{\partial v} + (x+t)\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \frac{\partial}{\partial u}$
e.	$B(u)=\frac{1}{1+u^2}\exp(m\arctan u)$, $K(u)=\sqrt{1+u^2}\exp(n\arctan u)$	$\mathbf{X}_4 = ((m-n)v+x)\frac{\partial}{\partial v} + (-v+(m-n)x)\frac{\partial}{\partial x}$ $+(m-2n)t\frac{\partial}{\partial t} + (1+u^2)\frac{\partial}{\partial u}$
Case 5 Nonlinear diffusion, linear convection		
a.	$B(u)=u^m$, $K(u)=0$ $m\neq 0,-2$	$\mathbf{X}_4 = v\frac{\partial}{\partial v} + x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}$ $\mathbf{X}_5 = (m+2)v\frac{\partial}{\partial v} + (m+1)x\frac{\partial}{\partial x}$ $+(m+2)t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$
b.	$B(u)=\mathrm{e}^{u}$, $K(u)=0$	$\mathbf{X}_4 = v\frac{\partial}{\partial v} + x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}$ $\mathbf{X}_5 = (v+x)\frac{\partial}{\partial v} + x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$
Case 7 (includes 8, 9) Linear diffusion, nonlinear convection		
a.	$B(u)=1$, $K(u)=u^n$ $n\neq 0,1,2$	$\mathbf{X}_4 = (-n+2)v\frac{\partial}{\partial v} + (-n+1)x\frac{\partial}{\partial x}$ $+(-2n+2)t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$
b.	$B(u)=1$, $K(u)=u\log u$	$\mathbf{X}_4 = v\frac{\partial}{\partial v} + t\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}$
c.	$B(u)=1$, $K(u)=\mathrm{e}^{u}$	$\mathbf{X}_4 = (x-v)\frac{\partial}{\partial v} - x\frac{\partial}{\partial x} - 2\frac{\partial}{\partial t} + \frac{\partial}{\partial u}$
Case 10, 11 Linear diffusion, specific convection forms		
a.	$B(u)=1$, $K(u)=0$	(Linear heat equation)
b.	$B(u)=1$, $K(u)=u^2$	(Burgers equation)

All equations admit the symmetries $\mathbf{X}_1 = \frac{\partial}{\partial x}$, $\mathbf{X}_2 = \frac{\partial}{\partial t}$, $\mathbf{X}_3 = \frac{\partial}{\partial v}$; Cases 1, 3, 6 admit only these and are not shown. Case numbering is as per invariant `rifsimp`. The symmetry groups for Cases 10, 11 are not shown: see [1, Sects. 4.2.4, 7.2.1] for details

Conversely a very well-chosen ranking may lead to all pivots being invariant, in which case our method is merely able to confirm this fact.

We emphasise that all the theory presented in this paper is directed towards verifying the usual infinitesimal criterion of invariance of DEs [1, Sect. 4.3.1], [2, Sect. 2.3], [3, Sect. 5.3]. If the DEs are not locally solvable [2, Sect. 2.6], then it is possible that some symmetries are missed. Thus, case splits detected as invariant by our method are definitely invariant; while those that fail the invariance test should be regarded as 'not known to be invariant'. Similar limitations apply if a 0th order case splitting equation arises, since infinitesimal methods are blocked in that case.

Regardless of these (usual) caveats, the methods described in this paper are a useful adjunct to algorithmic symmetry classification methods using DRC. For DEs containing a few arbitrary functions of one variable, such DRC methods can be very successful. Our invariance checking can be used to gauge the quality of a classification produced using `rifsimp` and to guide it towards improved classifications. Because of the way our calculations are organised, the computational cost is modest so long as expression swell is not overwhelming `rifsimp`.

Finally we remark that the problem of finding conservation laws for a class of DEs has many features in common with the symmetry classification problem. The equivalence group acts in a similar way (see e.g. [36]) and it should be possible to adapt our method to this case.

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